UNCLASSIFIED

AD **400 687**

Reproduced
by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

63-3-1

AD

WATERTOWN ARSENAL LABORATORIES

CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S EXTENSION PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS

TECHNICAL REPORT WAL TR 811.8/4

BY

OSCAR L. BOWIE

ASTIA

APR 10 1963

TISIA

DATE OF ISSUE - MARCH 1963
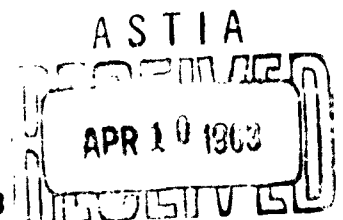
WATERTOWN ARSENAL
WATERTOWN 72, MASS.

CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S EXTENSION
PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS

Technical Report WAL TR 811.8/4

By

Oscar L. Bowie

Date of Issue - March 1963

WATERTOWN ARSENAL
WATERTOWN 72, MASS.

WATERTOWN ARSENAL LABORATORIES

TITLE

CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S EXTENSION
PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS

ABSTRACT

Muskhelishvili's extension principle for the case of geometries in-
volving corners and cracks and hence multivalued mapping functions is con-
sidered. It is shown that a useful application of the procedure can be
found in the study of the local stress distribution near the base of a
crack. The structures of the stress functions on the Riemann surface
defining the mapping function in the neighborhood of σ_0 corresponding to
the crack root can be found and the local stress distribution shown to
depend on ϕ' (σ_0).

OSCAR L. BOWIE
MATHEMATICIAN

APPROVED:

J. F. SULLIVAN
Director
Watertown Arsenal Laboratories

I. INTRODUCTION

In the plane theory of elasticity, Muskhelishvili's[1] formulation of the problem in terms of a single complex function defined in both the occupied region and its complement in the plane is well-known. Applications of the concept to regions defined by polynomial and rational mappings of several basic regions have been carried out, e.g., mappings of the unit circle by Kartsivadze[2]. The author[3] recently investigated the extension of the concept to multivalued mapping functions of the type encountered in the description of boundary corners.

This report is intended to show a practical application of the extension principle when applied to multivalued mapping functions. In the study of the local stress distribution near the base of a crack, it will be shown that the extension argument provides necessary information as to the character of the stress function. A simple conversion of the stress function defined in the auxiliary complex plane to the stress concentration in the physical plane is thus provided.

II. THE EXTENSION PRINCIPLE AND MULTIVALUED MAPPINGS

For concreteness, we consider regions in the physical Z-plane corresponding to a closed contour C and its interior S_Z^+. It will be assumed that a mapping relation

$$Z = \omega(\zeta) \tag{1}$$

exists which maps conformally the interior of the unit circle in the ζ-plane, S_ζ^+, into S_Z^+ and carries the unit circle τ into C. The existence of corner points in C implies branch point type of singularities in equation 1 lying on the unit circle, e.g., consider the Schwartz-Christoffel transformation.

Muskhelishvili's original formulation depends on representing the stresses and displacements in terms of two complex functions $\phi(\zeta)$ and $\psi(\zeta)$ holomorphic in S_ζ^+ as

$$\sigma_y + \sigma_x = 2R\ell[\phi'(\zeta)/\omega'(\zeta)], \tag{2}$$

$$\sigma_y - \sigma_x + 2i\,\tau_{xy} = 2\{\overline{\omega(\zeta)}[\phi'(\zeta)/\omega'(\zeta)]' + \psi'(\zeta)\}/\omega'(\zeta), \tag{3}$$

$$2\mu(u + iv) = \mathcal{K}\phi(\zeta) - \omega(\zeta)\overline{\phi'(\zeta)}/\overline{\omega'(\zeta)} - \overline{\psi(\zeta)}, \tag{4}$$

where primes denote differentiation with respect to ζ, bars denote complex conjugates, and μ and $\mathcal{K}$ are material constants. For boundary conditions in terms of the applied load on C, the corresponding conditions on the unit circle $\zeta = \sigma = e^{i\alpha}$ can be written as

$$\phi(\sigma) + \omega(\sigma)\overline{\phi'(\sigma)/\omega'(\sigma)} + \overline{\psi(\sigma)} = g(\sigma), \tag{5}$$

where $g(\sigma)$ is prescribed by the mapping and the given applied load.

The stress analysis requires the determination of the two functions $\phi(\zeta)$ and $\psi(\zeta)$ which must be analytic in S_ζ^+ and obey the condition, equation 5, on the unit circle. Muskhelishvili's extension principle consists essentially in replacing $\psi(\zeta)$ by an extended definition of $\phi(\zeta)$. Using the same form of extension as Kartzivadze, we define $\phi(\zeta)$ in the region exterior to the unit circle, S_ζ^-, as

$$\phi(\zeta) = -\omega(\zeta)\overline{\phi}'(1/\zeta)/\overline{\omega}'(1/\zeta) - \overline{\psi}(1/\zeta), \quad \zeta \in S_\zeta^-, \tag{6}$$

where

$$\overline{f}(1/\zeta) = \overline{f(1/\overline{\zeta})}. \tag{7}$$

Clearly $\phi(\zeta)$ as defined by equation 6 is analytic in S_ζ^- with the possible exception of the point at infinity. It follows from equation 6 that

$$\psi(\zeta) = -\overline{\phi}(1/\zeta) - \overline{\omega}(1/\zeta)\phi'(\zeta)/\omega'(\zeta), \quad \zeta \in S_\zeta^+. \tag{8}$$

It is evident for the mapping class considered that $\psi(\zeta)$ as defined by equation 8 is analytic in S_ζ^+ with the possible exception of $\zeta = 0$.

One of the conditions on the extended definition of $\phi(\zeta)$ is therefore the insurance of the analyticity of $\psi(\zeta)$ at $\zeta = 0$ in equation 8. The remaining condition arises from the satisfaction of the boundary condition equation 5. Replacing $\overline{\psi(\sigma)}$ in equation 5 by the definition equation 8, we find

$$\phi^+(\sigma) - \phi^-(\sigma) + [\omega^+(\sigma) - \omega^-(\sigma)]\overline{\phi'^-(\sigma)}/\overline{\omega'^-(\sigma)} = g(\sigma), \tag{9}$$

where the notation $f^-(\sigma)$, $[f^+(\sigma)]$, is defined as the value of $f(\zeta)$ as $\zeta \to \sigma$ through S_ζ^-, $[S_\zeta^+]$, respectively.

For polynomial mapping functions, equation 9 reduces to the so-called problem of linear relationship, i.e.

$$\phi^+(\sigma) - \phi^-(\sigma) = g(\sigma) \tag{10}$$

and the solution can be found in terms of a polynomial and a Cauchy integral.

When branch points in the mapping function lie on the unit circle the simplification equation 10 is no longer possible. The mapping function equation 1 must be considered as defined on an appropriate Riemann surface. With little loss in generality, we assume the appropriate branch of the mapping function lies on the top sheet and the branch cuts are intervals of the unit circle joining the branch points. It is evident that across the branch cuts, $\omega^+(\sigma) - \omega^-(\sigma) \neq 0$ and hence the simplification equation 10 is no longer possible.

Although the effectiveness of the extension principle in the case of multivalued mapping functions as a numerical procedure has little advantage over other techniques, one can argue that considerable insight into the

character of the stress functions is gained. A practical use of such insight will now be demonstrated in the study of the local stress distribution near the base of cracks occurring in geometries involving corner points.

III. AN APPLICATION OF THE EXTENSION PRINCIPLE

Let us assume that part of C corresponds to a crack; thus, there corresponds an interval τ_c on the unit circle which maps into the crack. Let σ_0 denote the point on τ_c which maps into the tip of the crack Z_0. Then

$$\omega'(\zeta) = (\zeta - \sigma_0)W(\zeta), \tag{11}$$

where $W(\zeta)$ is analytic in S_ζ^+ and nonvanishing in S_ζ^+ and τ_c. A finite number of branch points are assumed on the unit circle necessary to the description of the corner points of C.

Since C is a continuous closed curve, the branch cuts can be taken as intervals of the unit circle. Furthermore, it is easy to show that no cut is necessary on τ_c. By a rotation, the crack can be considered as falling on the real axis. Then $\varpi(1/\zeta)$ is the analytic continuation of $\omega(\zeta)$ across τ_c by Schwartz's reflection theorem. Since $\omega(\zeta)$ can be continued analytically across τ_c, certainly $\omega(\zeta)$ is analytic at $\zeta = \sigma_0$. Noting that $\omega'(\sigma_0) = 0$, it follows that in the neighborhood of σ_0,

$$Z - Z_0 = \sum_{n=2}^{\infty} S_n(\sigma - \sigma_0)^n. \tag{12}$$

If polar coordinates (r, θ) are introduced by

$$Z - Z_0 = re^{i\theta} \tag{13}$$

then by reversion of series,

$$\zeta - \sigma_0 \approx \sqrt{(Z - Z_0)/S_2} = \sqrt{r/S_2}\, e^{i\theta/2}. \tag{14}$$

From equations 2 and 3 it is clear that an estimate of the stresses requires a knowledge of the local behavior of the stress functions. For simplicity, we assume the crack surfaces are free from applied load. Then it is possible to define $g(\sigma)$ so that $g(\sigma) = 0$ on τ_c. From equation 9 it is clear that $\phi^+(\sigma) = \phi^-(\sigma)$ on τ_c for this case. Thus, $\phi(\zeta)$ as defined by equation 6 is the analytic continuation of $\phi(\zeta)$, $\zeta \in S_\zeta^+$, across the interval τ_c (in the top sheet of the Riemann surface for the mapping function). In particular, $\phi(\zeta)$ is analytic at $\zeta = \sigma_0$, thus,

$$\phi(\zeta) = \phi'(\sigma_0) + \phi''(\sigma_0)(\zeta - \sigma) + - - - \tag{15}$$

is valid in a neighborhood of σ_0 overlapping points in S_ζ^+ and S_ζ^-. Thus, it follows easily that

$$\sigma_x + \sigma_y \approx R\ell \left| \frac{2\sqrt{2} \; [\phi'(\sigma_0)]e^{-i\theta/2}}{\sqrt{\omega''(\sigma_0)} \; \sqrt{r}} \right|. \tag{16}$$

To approximate equation 3, the structure of $\psi(\zeta)$ is necessary. From the preceding arguments, $\phi'(\zeta)/\omega'(\zeta)$ is analytic at σ_0 except for a simple pole, thus,

$$\phi'(\zeta)/\omega'(\zeta) = a_{-1}(\zeta - \sigma_0)^{-1} + a_0 + a_1(\zeta - \sigma_0) + - - - . \tag{17}$$

Since $\phi(\zeta)$ and $\omega(\zeta)$ are analytic at σ_0 then so are $\bar{\phi}(1/\zeta)$ and $\bar{\omega}(1/\zeta)$; thus, from equation 8, $\psi(\zeta)$ has a simple pole at σ_0. We can write

$$\psi(\zeta) = A_{-1}(\zeta - \sigma_0)^{-1} + A_0 + A_1(\zeta - \sigma_c) + - - - . \tag{18}$$

Comparison of equations 8 and 18 and the series expansions above yields

$$A_{-1} = -a_{-1}\bar{Z}_0 = -\phi'(\sigma_0)\bar{Z}_0/2S_2,$$

$$A_0 = -\overline{\phi(\sigma_0)} - a_0 \bar{Z}_0, \tag{19}$$

$$A_1 = \overline{\phi'(\sigma_0)}/\sigma_0^2 - \bar{S}_2 \, a_{-1}/\sigma_0^4 - \bar{Z}_0 a_1.$$

It is then a straightforward matter from equation 3 to show

$$\sigma_y - \sigma_x + 2i \; \tau_{xy} \approx \frac{\sqrt{2} \; e^{-i\theta/2}}{\sqrt{r\omega''(\sigma_0)}} \left[\frac{\overline{\phi'(\sigma_0)}}{\sigma_0^2} - \frac{\phi'(\sigma_0)}{2\sigma_0^4} - \frac{\phi'(\sigma_0)e^{-2i\theta}}{2} \right]. \tag{20}$$

OBSERVATIONS

It was shown in the preceding section that the stresses in the vicinity of an unloaded crack can be approximated by equations 16 and 20. Since $\omega''(\sigma_0)$ can be calculated directly from the mapping function, the key computation is the evaluation of $\phi'(\sigma_0)$. Thus, for geometries involving corner points and cracks, the extension argument can be used to show that the local stresses in the crack vicinity depend on the evaluation of $\phi'(\sigma_0)$, independent of the procedure adopted, e.g., polynomial approximation, power series, integral equations, etc.

Generalizations of the arguments used in Section III to a larger class of geometries are self-evident. Furthermore, the restriction of the loading conditions on the crack surface can be removed by a modification of the argument. In particular, if $g(\sigma) \neq 0$ on the interval corresponding to the crack surface,

$$\phi(\zeta) = \phi_1(\zeta) + \frac{1}{2\pi i} \int_{\tau_c} \frac{g(t)dt}{(t - \zeta)} , \qquad (21)$$

where $\phi_1(\zeta)$ is analytic in the neighborhood of σ_0. Thus, the character of $\phi(\zeta)$ and hence $\psi(\zeta)$ in the vicinity of σ_0 is known and an argument similar to that of Section III can again be carried out.

REFERENCES

1. MUSKHELISHVILI, N. I., Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff Ltd., Groningen, Holland, 1953.

2. KARTZIVADZE, I. N., The Fundamental Problems of the Theory of Elasticity for the Elastic Circle, Comptes rendus de l'académie des sciences de l'U.R.SS. 20. v. XII, 1943, p. 95-104.

3. BOWIE, O. L., Boundary Irregularities in Plane Elasticity, Watertown Arsenal Laboratories, WAL TR 893.6/2, December 1959.

Report No.: WAL TR 811.8/4   Title: Crack Stresses: Application of
            March 1963   Muskhelishvili's Extension Principle
                                      for Multivalued Mapping Functions

---

Distribution List approved by 1st Indorsement from Ordnance Weapons
Command, ORDOW-TB, dated 4 January 1962

No. of Copies	TO
	Office of the Director of Defense Research and Engineering, Room 3D-1067, The Pentagon, Washington 25, D. C.
1	ATTN: Mr. J. C. Barrett
	Commander, Armed Services Technical Information Agency, Arlington Hall Station, Arlington 12, Virginia
10	ATTN: TIPDR
	Advanced Research Project Agency, The Pentagon, Washington 25, D.C.
1	ATTN: Dr. G. Mock
1	Solid Propellant Information Agency, Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Maryland
	Commanding General, U. S. Army Materiel Command, Washington 25, D. C.
1	ATTN: AMCRD-RS
	Commanding General, U. S. Army Missile Command, Redstone Arsenal, Alabama
2	ATTN: AMSMI-RB, Redstone Scientific Information Center
1	AMSMI-RRS, Mr. R. E. Ely
1	AMSMI-RKX, Mr. R. Fink
1	AMSMI, Mr. W. K. Thomas
1	AMSMI-RSM, Mr. E. J. Wheelahan
1	Commanding General, U. S. Army Mobility Command, Detroit 9, Michigan
1	Commanding General, U. S. Army Munitions Command, Dover, New Jersey
	Commanding General, U. S. Army Test and Evaluation Command, Aberdeen Proving Ground, Maryland
2	ATTN: AMSTE, Technical Library

No. of Copies	TO

Commanding General, U. S. Army Weapons Command,
Rock Island, Illinois

1   ATTN: AMSWE-IX, Industrial Division
1   AMSWE-TX, Research Division
1   AMSWE-IM, Industrial Mobilization Branch
1   AMSWE-GU, Security Officer

Commanding Officer, Diamond Ordnance Fuze Laboratory,
Connecticut Avenue and Van Ness Street, N.W., Washington 25, D. C.

1   ATTN: AMCDO-TIB

Commanding Officer, Frankford Arsenal, Philadelphia 37, Pennsylvania

1   ATTN: SMUFA-1330
1   SMUFA-0270, Library

Commanding Officer, Picatinny Arsenal, Dover, New Jersey

1   ATTN: AMSMU, Mr. J. J. Scavuzzo, Plastics and Packaging Laboratory

1   Commanding Officer, PLASTEC, Picatinny Arsenal, Dover, New Jersey

Commanding Officer, Springfield Armory, Springfield 1, Massachusetts

1   ATTN: SWESP-TX, Research and Development Division

Commanding Officer, Watertown Arsenal, Watertown 72, Massachusetts

1   ATTN: SMIWT-EX, Chief, Engineering Division
1   SMIWT-OE, Industrial Engineering Section

Commanding Officer, Watervliet Arsenal, Watervliet, New York

1   ATTN: SWEWV-RR

Commanding General, U. S. Army Chemical Warfare Laboratories,
Army Chemical Center, Maryland

1   ATTN: Technical Library

1   Commanding Officer, U. S. Army Environmental Health Laboratory,
Army Chemical Center, Maryland

Commandong Officer, Engineering Research and Development Laboratory,
Fort Belvoir, Virginia

1   ATTN: Materials Branch

Commanding General, Quartermaster Research and Development Command,
Natick, Massachusetts

1   ATTN: AMXRC, Clothing and Organic Materials Division

No. of Copies	TO

Headquarters, U. S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey
1   ATTN: Materials Branch

1   Director, Army Research Office, Office Chief Research and Development, U. S. Army, The Pentagon, Washington 25, D. C.

1   Commanding Officer, U. S. Army Research Office (Durham), Box CM, Duke Station, Durham, North Carolina

Chief of Research and Development, U. S. Army Research and Development Liaison Group, APO 757, New York
1   ATTN; Dr. B. Stein

Chief, Bureau of Naval Weapons, Department of the Navy, Room 2225, Munitions Building, Washington 25, D. C.
1   ATTN: RMMP

Chief, Bureau of Ships, Department of the Navy, Washington 25, D.C.
1   ATTN: Code 344

Chief, Office of Naval Research, Department of the Navy, Washington 25, D. C.
1   ATTN: Code 423

Chief, Special Projects Office, Bureau of Naval Weapons, Department of the Navy, Washington 25, D. C.
1 ___ ATTN: SP 271

Commander, U. S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland
1   ATTN: WM

Commander, U. S. Naval Ordnance Test Station, China Lake, California
1   ATTN: Technical Library Branch

Commander, U. S. Naval Research Laboratory, Anacostia Station, Washington 25, D. C.
1   ATTN: Technical Information Center

U. S. Air Force Directorate of Research and Development, Room 4D-313, The Pentagon, Washington 25, D. C.
1   ATTN: Lt. Col. F. B. Shipp, Jr.

ARDC Flight Test Center, Edwards Air Force Base, California
1   ATTN: Solid Systems Division, FTRSC

No. of Copies	TO
	AMC Aeronautical Systems Center, Wright-Patterson Air Force Base, Ohio
2	ATTN: Manufacturing and Materials Technology Division, LMBMO
	National Aeronautics and Space Administration, 1520 H Street, N.W., Washington 25, D. C.
1	ATTN: Mr. G. C. Deutsch
1	Mr. R. V. Rhode
	Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, California
1	ATTN: Dr. L. Jaffe
	George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama
1	ATTN: M-S&M-M
1	M-F&AE-M
	Commanding Officer, U. S. Army Materials Research Agency, Watertown 72, Massachusetts
5	ATTN: AMXMR-LXM, Technical Information Section
1	AMXMR-OPT, Office of Programs and Technology
1	AMXMR, Dr. R. Beeuwkes, Jr.
1	Author

69 -- TOTAL COPIES DISTRIBUTED

AD_____ Accession No.
Watertown Arsenal Laboratories, Watertown 72, Mass.
CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S
EXTENSION PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS -
Oscar L. Bowie

Report No. WAL TR 811.8/4, March 1963, 8 pp, OMS Code
5011.11.838, D/A Project 59925001, Unclassified Report

Muskhelishvili's extension principle for the case of
geometrics involving corners and cracks and hence
multivalued mapping functions is considered. It is
shown that a useful application of the procedure can be
found in the study of the local stress distribution
near the base of a crack. The structures of the stress
functions on the Riemann surface defining the mapping
function in the neighborhood of σ_0 corresponding to the
crack root can be found and the local stress distribu-
tion shown to depend on $\phi(\sigma_0)$.

NO DISTRIBUTION LIMITATIONS

AD_____ Accession No.
Watertown Arsenal Laboratories, Watertown 72, Mass.
CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S
EXTENSION PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS -
Oscar L. Bowie

Report No. WAL TR 811.8/4, March 1963, 8 pp, OMS Code
5011.11.838, D/A Project 59925001, Unclassified Report

Muskhelishvili's extension principle for the case of
geometrics involving corners and cracks and hence
multivalued mapping functions is considered. It is
shown that a useful application of the procedure can be
found in the study of the local stress distribution
near the base of a crack. The structures of the stress
functions on the Riemann surface defining the mapping
function in the neighborhood of σ_0 corresponding to the
crack root can be found and the local stress distribu-
tion shown to depend on $\phi(\sigma_0)$.

NO DISTRIBUTION LIMITATIONS

AD_____ Accession No.
Watertown Arsenal Laboratories, Watertown 72, Mass.
CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S
EXTENSION PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS -
Oscar L. Bowie

Report No. WAL TR 811.8/4, March 1963, 8 pp, OMS Code
5011.11.838, D/A Project 59925001, Unclassified Report

Muskhelishvili's extension principle for the case of
geometrics involving corners and cracks and hence
multivalued mapping functions is considered. It is
shown that a useful application of the procedure can be
found in the study of the local stress distribution
near the base of a crack. The structures of the stress
functions on the Riemann surface defining the mapping
function in the neighborhood of σ_0 corresponding to the
crack root can be found and the local stress distribu-
tion shown to depend on $\phi(\sigma_0)$.

NO DISTRIBUTION LIMITATIONS

AD_____ Accession No.
Watertown Arsenal Laboratories, Watertown 72, Mass.
CRACK STRESSES: AN APPLICATION OF MUSKHELISHVILI'S
EXTENSION PRINCIPLE FOR MULTIVALUED MAPPING FUNCTIONS -
Oscar L. Bowie

Report No. WAL TR 811.8/4, March 1963, 8 pp, OMS Code
5011.11.838, D/A Project 59925001, Unclassified Report

Muskhelishvili's extension principle for the case of
geometrics involving corners and cracks and hence
multivalued mapping functions is considered. It is
shown that a useful application of the procedure can be
found in the study of the local stress distribution
near the base of a crack. The structures of the stress
functions on the Riemann surface defining the mapping
function in the neighborhood of σ_0 corresponding to the
crack root can be found and the local stress distribu-
tion shown to depend on $\phi(\sigma_0)$.

NO DISTRIBUTION LIMITATIONS